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FUSS-NARAYANA STATISTICS

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Abstract. We show that valleys, high peaks, and modular ascents are statistics of Fuss-Catalan paths having a distribution given by the Fuss-Narayana number. We prove the results using the Cycle Lemma and provide bijections among them. We also show that relative peaks are independent of the base path. In particular, valleys and high peaks can be obtained from relative peaks by fixing the base path in certain ways.

1. Introduction

A Dyck path of length n is a lattice path from $(0,0)$ to (n,n) using east steps $E=(1,0)$ and north steps $N=(0,1)$ such that it stays weakly above the diagonal line $y=x$. It is well-known that the number of Dyck paths of length n is given by the famous Catalan numbers

$$\frac{1}{n+1}\binom{2n}{n}.$$

One of the most common refinements of the Catalan numbers is given by the Narayana numbers

$$N(n,k)=\frac{1}{n}\binom{n}{k}\binom{n}{k+1}.$$

The statistic on the set of all Dyck paths having a distribution given by the Narayana numbers is called a *Narayana statistic*. Some well-known Narayana statistics for a Dyck path P are:

1. $\mathrm{va}(P)$: the number of *valleys* (sequences EN);
2. $\mathrm{hp}(P)$: the number of *high peaks* (sequences NE strictly above the line $y=x$);

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3. $\mathrm{ea}(P)$: the number of *even ascents*, i.e., the number of N's in an even position;
4. $\mathrm{lnfs}(P)$: the number of *long non-final sequences*, more precisely the number of sequences NNE and EEN.

Narayana distributions are explained in [6].

An s-Fuss-Catalan path of length n is a path from $(0,0)$ to (n, sn) using east steps E and north steps N such that it stays weakly above the line $y = sx$. The number of s-Fuss-Catalan paths of length n is given by the s-Fuss-Catalan numbers

$$\frac{1}{sn+1}\binom{(s+1)n}{n}.$$

In this article, we extend Narayana statistics to s-Fuss-Catalan paths. More precisely, we show that the following statistics of a Fuss-Catalan path P have distributions given by the s-Fuss-Narayana numbers

$$\frac{1}{n}\binom{sn}{k}\binom{n}{k+1}:$$

1. $\mathrm{va}(P)$: the number of *valleys* (sequences EN);
2. $\mathrm{hp}(P)$: the number of *high peaks* (sequences NE strictly above the line $y = sx$);
3. $\mathrm{ma}(P)$: the number of *modular ascents*, i.e., the number of N's in an $i(s+1)$th position in P for $i = 1, 2, \ldots, n$.

Brändén [1] defined the *descent set* $\mathrm{des}_Q(P)$ *of a Dyck path P with respect to a fixed Dyck path Q* and showed that va is obtained when a fixed Dyck path is $N \cdots NE \cdots E$ and hp is obtained when a fixed Dyck path is $NENE \cdots NE$. We will use the notion of *relative peaks* $\mathrm{rp}_Q(P)$ instead of $\mathrm{des}_Q(P)$. We also generalize relative peaks to the case of Fuss-Catalan paths and give a simple direct proof to show that relative peaks are independent of the base path.

2. Relative peaks

In this section, we show that va and hp are Fuss-Narayana statistics. Moreover, we show that they are examples of relative peaks. We also show that relative peaks are independent of the base Fuss-Catalan path.

The proofs of some theorems in this and the next section will rest on the "Cycle Lemma" of Dvoretzky and Motzkin [3]. Let $\mathcal{A}$ denote a set of alphabets and let $\mathcal{A}^*$ be the set of all words generated by $\mathcal{A}$. The *weight function* is the map $\sigma : \mathcal{A}^* \to (\mathbb{Z}, +)$ induced by the weight function $\sigma : \mathcal{A} \to (\mathbb{Z}, +)$ on $\mathcal{A}$. Given any word $w = w_1 w_2 \cdots w_n \in \mathcal{A}^*$, a *conjugate* of w is an element of $\mathcal{A}^*$ of the form $w_i w_{i+1} \cdots w_n w_1 w_2 \cdots w_{i-1}$ for some $1 \leq i \leq n$. Then the Cycle Lemma can be stated in the following form.

Lemma 2.1 (Cycle Lemma). *For $w \in \mathcal{A}^*$ with $\sigma(w) = 1$, there is unique conjugate of w such that all of its nonempty prefixes have positive weight.*

The following theorem is given by Cigler [2]. We provide a proof using the Cycle Lemma. The proof is a straightforward generalization of the proof for Dyck paths given in [4, Theorem 2.5.2 (6)].

Theorem 2.2. *The number of s-Fuss-Catalan paths with k valleys is the s-Fuss-Narayana number.*

Proof. Since any s-Fuss-Catalan path with k valleys has $k+1$ peaks, it is enough to show that the number of s-Fuss-Catalan paths with $k+1$ peaks is the s-Fuss-Narayana number. If we consider paths from $(0, -1)$ to (n, sn) that end with an east step E with $k+1$ peaks NE, each one has n conjugates of this form. Since we can write such a path as $E^{i_0 - 1} N^{j_1} E^{i_1} N^{j_2} E^{i_2} \cdots N^{j_{k+1}} E^{i_{k+1}}$ where

$$(i_0 - 1) + i_1 + \cdots + i_{k+1} = n; \qquad i_l > 0$$

and

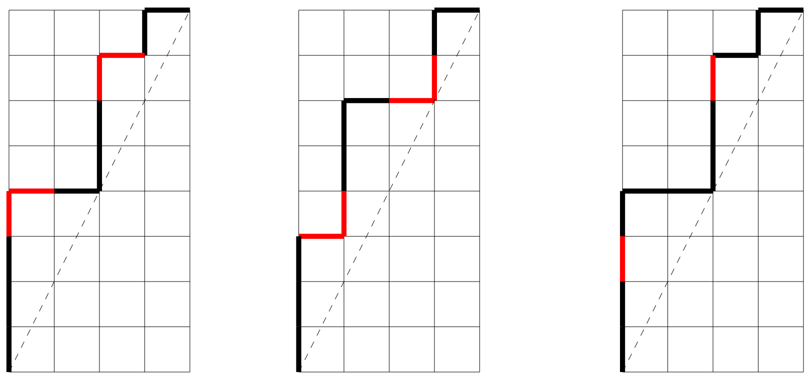
$$j_1 + \cdots + j_{k+1} = sn + 1; \qquad j_l > 0,$$

there are $\binom{sn}{k}\binom{n}{k+1}$ such paths. If we consider $\mathcal{A} = \{N^i E : i \geq 0\}$ with the weight function $\sigma(N^i E) = i - s$, Lemma 2.1 implies that each path has a unique conjugate such that all of its nonempty prefixes have positive weight. The result follows since a path such that all of its nonempty prefixes have positive weight begins with at least $s+1$ north steps and stays weakly above the line $y = sx$ except its first north step. $\square$

The bijection ϕ is defined for Dyck paths by Sulanke [5] and it works for Fuss-Catalan paths as well.

Theorem 2.3. *The number of s-Fuss-Catalan paths with k high peaks is the s-Fuss-Narayana number.*

Proof. Define the map ϕ from the set $\mathcal{P}_1$ of s-Fuss-Catalan paths with k high peaks to the set $\mathcal{P}_2$ of s-Fuss-Catalan paths with k valleys as follows: If (x, y) is the vertex of a high peak of a path P in $\mathcal{P}_1$, then



(a) P_1 (high peaks) (b) P_2 (valleys) (c) P_3 (modular ascents)

FIGURE 1. 2-Fuss-Catalan paths of length 4

$(x+1, y-1)$ will be the vertex of a valley of $\phi(P)$. Since the set of valleys completely determines the Fuss-Catalan path, $\phi(P)$ is contained in $\mathcal{P}_2$. Since the set of high peaks completely determines the Fuss-Catalan path by adding some peaks touching the line $y = sx$, ϕ^{-1} is also well-defined. □

EXAMPLE 2.4. *The path P_1 with two high peaks and its corresponding path P_2 with two valleys are shown in Figures 1(a) and 1(b). High peaks in P_1 and valleys in P_2 are colored in red.*

We can write the $(i+1)$st point x in an s-Fuss-Catalan path $P = p_1 p_2 \cdots p_{(s+1)n}$ as $x = p_1 + p_2 + \cdots + p_i$ where $p_j = (0,1)$ or $p_j = (1,0)$ for $1 \leq j \leq i \leq (s+1)n$. We say that a point $x = p_1 + p_2 + \cdots + p_i$ in an s-Fuss-Catalan path $P = p_1 p_2 \cdots p_{(s+1)n}$ is a *relative peak* of P with respect to a fixed s-Fuss-Catalan path Q if

- $p_i p_{i+1} = NE$ and x is strictly north-west of Q or
- $p_i p_{i+1} = EN$ and x is strictly south-east of Q.

Sometimes we also say that the corresponding step NE or EN is a relative peak of P with respect to Q. For a fixed s-Fuss-Catalan path Q, the statistic $\mathrm{rp}_Q(P)$ of an s-Fuss-Catalan path P is defined as the number of relative peaks of P with respect to Q.

EXAMPLE 2.5. *Figure 2 shows the relative peaks of the* 2*-Fuss-Catalan path $P = NNNNNEENNENE$ (black and red) with respect to the path $Q = NNNENNNENNEE$ (blue). The points* $(0,5), (2,5)$*, and*

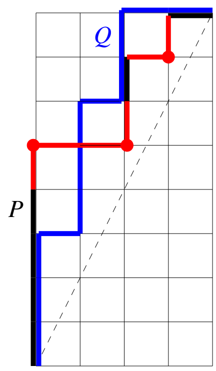


FIGURE 2. The relative peaks of P with respect to Q

$(3, 7)$ *are relative peaks of* P *with respect to* Q *and they are colored in red.*

EXAMPLE 2.6. *The statistics* va *and* hp *arise when fixing the base path* Q *in certain ways.*
(a) If $Q = NN \cdots NEE \cdots E$, *then* $\mathrm{rp}_Q = \mathrm{va}$.
(b) If $Q = N^sEN^sE \cdots N^sE$, *then* $\mathrm{rp}_Q = \mathrm{hp}$.

Two lattice paths Q_1 and Q_2 are said to be *adjacent* if they are differ by one box, i.e., $Q_1 = Q'NEQ''$ and $Q_2 = Q'ENQ''$ for some lattice paths Q' and Q''.

THEOREM 2.7. *The distributions of relative peaks of* s*-Fuss-Catalan paths are independent of the base path.*

Proof. Since any two s-Fuss-Catalan paths can be obtained from each other by a sequence of adjacent s-Fuss-Catalan paths, it is enough to show the case when two base paths are adjacent. Let $Q_1 = Q'NEQ''$ and $Q_2 = Q'ENQ''$ be adjacent s-Fuss-Catalan paths and let x be the end point of Q' and y be the beginning point of Q''. If an s-Fuss-Catalan path P does not pass through either x or y, we can see $\mathrm{rp}_{Q_1}(P) = \mathrm{rp}_{Q_2}(P)$. If an s-Fuss-Catalan path P_1 has the form $P'NEP''$ where x is the end point of P', then $\mathrm{rp}_{Q_1}(P_1) = \mathrm{rp}_{Q_2}(P_1) - 1$ since the end point of $P'N$ is a relative peak of P_1 with respect to Q_2 but is not a relative peak of P_1 with respect to Q_1. Also, $\mathrm{rp}_{Q_1}(P_2) = \mathrm{rp}_{Q_2}(P_2) + 1$ for the s-Fuss-Catalan path $P_2 = P'ENP''$ (where P' and P'' are equal to those in $P_1 = P'NEP''$) since the end point of $P'E$ is a relative peak of P_2 with respect to Q_1 but is not a relative peak of P_2 with respect to Q_2. Thus the distributions of relative peaks are independent of the base path. $\square$

3. Modular ascents

In this section, we show that ma is a Fuss-Narayana statistic. We also provide a bijection between the set of s-Fuss-Catalan paths with k valleys and the set of s-Fuss-Catalan paths with k modular ascents.

Theorem 3.1. *The number of s-Fuss-Catalan paths with k modular ascents is the s-Fuss-Narayana number.*

Proof. In order to prove the theorem, we express the Fuss-Narayana number in the followng form:

$$\frac{1}{n}\binom{sn}{k}\binom{n}{k+1} = \frac{1}{k+1}\binom{sn}{k}\binom{n-1}{k}.$$

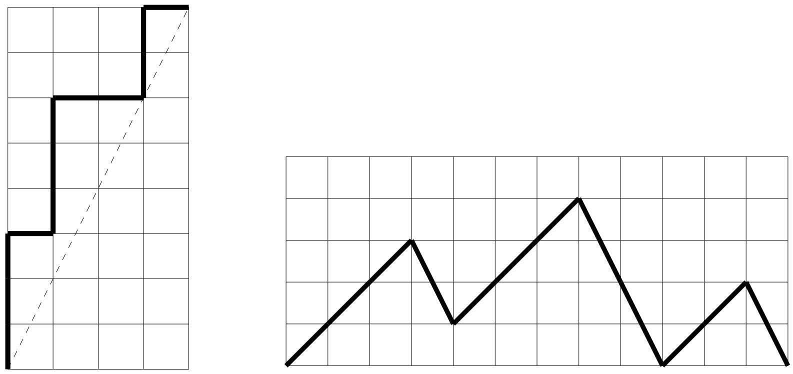
For convenience, we use the following interpretation of Fuss-Catalan paths in this proof. We replace a north step $N = (0,1)$ with an up step $U = (1,1)$ and an east step $E = (1,0)$ with a down step $D = (1,-s)$. Then an s-Fuss-Catalan path from $(0,0)$ to (n, sn) turns out to be a lattice path from $(0,0)$ to $((s+1)n, 0)$ such that it stays weakly above the x-axis. In Figure 3, (a) shows a 2-Fuss-Catalan path using N and E and (b) shows the same 2-Fuss-Catalan path using U and D.

Consider the set $\mathcal{P}$ of paths from $(0,0)$ to $((s+1)n, s+1)$ with $k+1$ modular up steps that end with an up step. Any path in $\mathcal{P}$ has $sn+1$ up steps and $n-1$ down steps. Since the paths end with an up step, we have k up steps to place in $n-1$ positions congruent modulo $s+1$ in $\binom{n-1}{k}$ ways. The remaining $sn-k$ up steps can be assigned in sn positions in $\binom{sn}{sn-k} = \binom{sn}{k}$ ways. Any path in $\mathcal{P}$ has $k+1$ conjugates that end with an up step.

In order to use the Cycle Lemma, we form an alphabet set

$$\mathcal{A} = \{w_1 w_2 \cdots w_i : i = (s+1)j \text{ for some } j \text{ and } w_i = U\}.$$

Note that each step in $\mathcal{A}$ goes up (or down) by a multiple of $s+1$. Thus we define the weight of a step in $\mathcal{A}$ by a difference of the heights of its beginning point and the end point divided by $s+1$. Lemma 2.1 implies that each path has a unique conjugate such that all of its nonempty prefixes have positive weight. Since such a path stays weakly above the



(a) (b)

FIGURE 3. Two representations of 2-Fuss-Catalan paths of length 4

x-axis, we get an s-Fuss-Catalan path by replacing the last up step with the down step and the result follows. $\square$

THEOREM 3.2. *There is a bijection between the set of s-Fuss-Catalan paths of length n with k valleys and the set of s-Fuss-Catalan paths of length n with k modular ascents.*

Proof. Let P be an s-Fuss-Catalan path of length n with k valleys. Let $(x_1, y_1), \ldots, (x_k, y_k)$ be the points of valleys of P. Define a sequence $m_1, m_2, \ldots, m_{n-1}$ by

$$m_i = \begin{cases} N & \text{if there is a valley whose } x\text{-coordinate is } i, \\ E & \text{otherwise,} \end{cases}$$

and $a_1, a_2, \ldots, a_{ns-1}$ by

$$a_i = \begin{cases} E & \text{if there is a valley whose } y\text{-coordinate is } i, \\ N & \text{otherwise.} \end{cases}$$

Define a map ψ by

$$\psi(P) = Na_1 \cdots a_{s-1} m_1 a_s \cdots a_{2s-1} m_2 \cdots m_{n-1} a_{(n-1)s} \cdots a_{ns-1} E.$$

Since all points of valleys of P lie weakly above the line $y = sx$, $\psi(P)$ is an s-Fuss-Catalan path of length n. Since exactly k of $m_1, \ldots, m_{n-1}$ are N, we can see that $\psi(P)$ has k modular ascents.

Conversely, given an s-Fuss-Catalan path Q of length n, we write

$$Q = Na_1 \cdots a_{s-1} m_1 a_s \cdots a_{2s-1} m_2 \cdots m_{n-1} a_{(n-1)s} \cdots a_{ns-1} E.$$

Then exactly k of $m_1, \ldots, m_{n-1}$ are N and exactly k of $a_1, \ldots, a_{ns-1}$ are E. If x_i is the position of ith N in the sequence $m_1, \ldots, m_{n-1}$ and y_i is the position of ith E in the sequence $a_1, \ldots, a_{ns-1}$, we define $\psi^{-1}(Q)$ to be the s-Fuss-Catalan path whose valleys are given by points $(x_1, y_1), \ldots, (x_k, y_k)$. □

Example 3.3. *As shown in Figure 1(b), $P_2 = NNNENNNEENNE$ is a 2-Fuss-Catalan path of length 4 with two valleys whose positions are $(1,3)$ and $(3,6)$. Thus we obtain sequences $m_1 = N, m_2 = E, m_3 = N$ and $a_1 = N, a_2 = N, a_3 = E, a_4 = N, a_5 = N, a_6 = E, a_7 = N$. Thus $\psi(P_2)$ is the lattice path $P_3 = NNNNEENNNENE$ shown in Figure 1(c). Modular ascents of P_3 are colored in red.*

4. Future work

Brändén [1] showed that the relative peaks for Dyck paths are independent of the base path using a bijection between the set of linear extensions of the poset $\mathbf{2} \times \mathbf{n}$ and the set of Dyck paths of length n. He also showed that the statistic lnfs is a Narayana statistics from the shelling of the order complex of the order ideals of the poset $\mathbf{2} \times \mathbf{n}$.

Although we prove that the relative peaks for Fuss-Catalan paths are independent of the base path, we could not find a poset whose linear extensions are in one-to-one correspondence with the set of all Fuss-Catalan paths. It would be nice if we could find such a poset and find a Fuss analogue of lnfs for Dyck paths.

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